Outcome-Independent Payoffs in Strategic Voting∗

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Abstract

We consider a game-theoretic model of voting in which players have an outcome-independent component to their preferences. This outcome-independent component—even if arbitrarily small—can dramatically affect the set of Nash equilibria in voting games because it determines how voters behave when they are not pivotal. Given incomplete information, some weak restrictions on voter preferences, and a sufficiently large number of voters, there is a unique Bayesian Nash equilibrium in which every player votes according to the outcome-independent component of his preferences. Our model helps explain (1) why people vote when participation is optional and voting is costly, and (2) why public and secret voting may lead to different outcomes.

∗We thank Professor Stephen Salant for his devoted attention, careful guidance, and invaluable suggestions. We also thank Professor Lones Smith, Professor Scott Page, and Professor Tilman Börgers for their helpful comments.
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1 Introduction

Game-theoretic models of voting typically assume that individuals vote in order to influence the outcome of an election. There is clear empirical evidence, however, that an individual voter rarely casts a deciding vote. For example, Mulligan and Hunter (2001) estimate the frequency of a pivotal vote to be about $2/v$, where $v$ is the total number of votes cast in an election. For elections with 1,000 votes cast, this puts the frequency of a pivotal vote at about 0.2%. For elections with at least 100,000 votes, which includes United States presidential elections, midterm elections, and elections for state-wide office, the frequency of a pivotal vote is less than 0.002%. If an individual voter has no power to influence the outcome of an election, why does he take the trouble to vote?

This paradox of voter participation motivates our model. What if individuals have some payoff from how they vote that is independent of the outcome their vote induces? Adding an “outcome-independent” component to voter preferences allows us to study interesting scenarios. For example, imagine a committee of voters where members care about not only the outcome of the vote, but also about the way in which their vote is perceived by other members, or perhaps by a committee chair. These outcome-independent payoffs may override committee members’ preferences over outcomes. Similarly, politicians have incentives to vote in a way that pleases their constituency, and may also respond to outcome-independent payoffs associated with vote trading or lobbying.

Our goal is to develop a model of voting behavior that accounts for these subtleties. We begin by decomposing agents’ utility functions into outcome-dependent and outcome-independent components. This decomposition permits a simple but interesting analysis of voter behavior. We show that an outcome-independent component dramatically affects the set of pure strategy Nash equilibria of a voting game. The intuition is quite simple: each non-pivotal voter cannot influence the vote’s outcome and must therefore vote in order to maximize the outcome-independent component of his payoff. We show that the only equilibrium in which some individual does not vote in accordance with the outcome-independent
component of his preferences is the knife-edge case in which there exist pivotal voters.

We then consider a game with incomplete information. Motivated by empirical evidence that voters are rarely pivotal in large elections, we study the case where the number of voters is large. We show that if voters are not certain to vote in any particular way, the probability any voter is pivotal tends to zero as the number of voters tends to infinity. This result and a few additional restrictions allow us to construct a voting game with incomplete information and to show that—given enough voters—there exists a unique Bayesian Nash equilibrium in which everyone votes according to the outcome-independent component of his preferences. We then use our model to help explain (1) why people vote when participation is optional and voting is costly, and (2) why public and secret voting may lead to different outcomes. We conclude by discussing some applications.

Our results demonstrate a deep lack of robustness in classical voting theory. In many contexts, one can reasonably expect voters to have an outcome-independent component to their preferences. Yet, even when arbitrarily small, outcome-independent payoffs can lead to large and systematic deviations from the predictions of common models of voting. In this paper, we present a simple and stylized model in order to emphasize the intuition behind this lack of robustness and its potential impact on strategic voting.

2 The Model

We consider the case where $N$ agents simultaneously submit votes over a finite set of alternatives, $\mathcal{V}$. We say that there is an outcome function $O : \mathcal{V}^N \rightarrow \sigma(\mathcal{V})$, where $\sigma(\mathcal{V})$ is a set of probability distributions over $\mathcal{V}$. We will eventually focus on the case where the outcome is decided by ‘plurality rule’. That is, if one outcome receives strictly more votes than any other, it is selected with probability 1 (ties are broken randomly and with equal probability). For now, however, we allow the outcome function to have a more general form. For example, any bill passed by Congress may be vetoed by the president, so that the outcome of a vote
in Congress is the resulting probability the bill in question becomes law. This probability may be some complicated function of the voting profile.

Suppose each voter has utility function \( u_i : \mathcal{V}^N \rightarrow \mathbb{R} \) which can be decomposed into the sum of two functions, \( u_i^O \) and \( f_i \). That is,

\[
  u_i(\bar{v}) = u_i^O(O(\bar{v})) + f_i(v_i)
\]

where \( v_i \) is player \( i \)'s vote and \( \bar{v} \) is the voting profile. Here, \( u_i^O \) represents the component of voter \( i \)'s utility which depends on the outcome, and \( f_i \) represents the component of \( i \)'s utility which is outcome-independent.

What motivates these preferences? Voters may respond to a number of economic and social payoffs. Individuals may vote for a policy or a candidate whose success directly or indirectly benefits them. This would generate preferences over votes dependent on the outcome. However, individuals may also vote to express their personal beliefs, to please their friends, or to upset their enemies. Or perhaps individuals vote in response to bribes or threats that do not depend on the vote’s outcome. These incentives generate preferences over votes independent of the outcome.

Note that outcome-dependent preferences and outcome-independent preferences do not necessarily conflict. In fact, we expect them to agree in many scenarios. However, decomposing utility functions this way permits a simple but interesting analysis of voter behavior.

To start, we make the following simple assumptions:

**Assumption 1.** There is complete information. That is, each player knows the payoffs and strategies available to other players.\(^1\)

**Assumption 2.** Each agent \( i \) is required to vote.\(^2\)

\(^1\)We relax Assumption 1 in Section 3.
\(^2\)We relax Assumption 2 in Section 4.
Assumption 3. Let $v_{-i}$ be the profile of votes cast by all players except player $i$. For any player $i$ and any fixed voting profile $v_{-i}$, the preferences over $\mathcal{V}$ induced by $u_i$ are strict. That is, we assume voters are never completely indifferent between voting for two alternatives. Also assume that the preferences over $\mathcal{V}$ induced by $u_i^O$ and $f_i$ are strict.

Definition 1. Given an outcome mapping $O : \mathcal{V}^n \rightarrow \sigma(\mathcal{V})$ and a voting profile $\bar{v}$, we say that an agent is pivotal if his vote can alter the outcome. That is, $i$ is pivotal if $O(v_{-i}, v_i) \neq O(v_{-i}, v'_i)$ for some $v_i, v'_i \in \mathcal{V}$.

Proposition 1. In any pure strategy Nash equilibrium, if some agent $i$ is not pivotal, then he must vote in order to maximize $f_i$.

Proof. If agent $i$ is not pivotal, $u_i^O(v_{-i}, v_i)$ is the same for all $v_i$. Therefore, for any $v_i, v'_i \in \mathcal{V}$, $u_i(v'_i) - u_i(v_i) = f_i(v'_i) - f_i(v_i)$ and $v$ maximizes $u_i$ if and only if $v$ maximizes $f_i$. \hfill \Box

Corollary 1. If when each agent votes to maximize $f_i$, there is a non-pivotal victory, then this outcome is a Nash equilibrium in pure strategies.

Corollary 2. If the outcome is decided by plurality rule, any equilibrium in which some agent $i$ does not vote according to $f_i$ has the following form: A group of pivotal agents vote for some victorious outcome and each non-pivotal agent votes according to $f_i$.

Example 1. Consider the following voting scenario with five voters. Suppose $\mathcal{V} = \{a, b, c\}$ and that each individual $i$ has a von Neumann-Morgenstern utility function $u_i = u_i^O + f_i$ where $u_i^O(a) = u_i^O(b) + 1 = u_i^O(c) + 2$ and $f_i(a) = f_i(b) - \epsilon = f_i(c) - 2\epsilon$, for arbitrarily small $\epsilon > 0$. In this case, each voter prefers $a$ to $b$ and $b$ to $c$ with respect to $u^O$, but each voter prefers $c$ to $b$ and $b$ to $a$ with respect to $f$. Let the outcome be decided by plurality rule and let ties be broken by assigning an equal probability of winning to each tied alternative. Then there are three pure strategy Nash equilibria\textsuperscript{3}:

\textsuperscript{3}To be precise, there are three types of pure strategy Nash equilibria (2 and 3 can have different permutations of agents voting for $a$, $b$, or $c$).
1. All five voters vote for c.

2. Three voters vote for a and two vote for c.

3. Three voters vote for b and two vote for c.

Note that in the first equilibrium, c is the least preferred outcome from the viewpoint of every voter. Yet, c wins unanimously. Every voter is rational and no voter is confused about the voting system. No voter is pivotal, so each simply responds to the outcome-independent component of his payoff. In the third equilibrium, like the first, no one votes for the outcome everyone most prefers.

Example 1 resembles an extreme case of the provision of a public good. Every voter is strictly better off if at least three individuals vote for alternative a. However, unless exactly two other individuals do so (so that a vote for alternative a by any of the other three voters would seal its victory), each player prefers to vote for alternative b or c.

Example 2. Suppose there are 100 voters and two alternatives (a and b), and suppose outcomes are decided by majority rule (ties are broken randomly and with equal probability). Assume exactly 50 of the 100 voters prefer to vote for a if they are pivotal (i.e. \( u_i^O(a) > u_i^O(b) \)) and 50 prefer to vote for b if they are pivotal. Also assume that among these 100 voters, \( K > 51 \) individuals prefer to vote for a if they are not pivotal (i.e. \( f_i(a) > f_i(b) \)) and 100\(-K\) prefer to vote for b if they are not pivotal. Then there is a pure strategy Nash equilibrium in which \( K \) individuals vote for a, and a wins. This equilibrium, however, is not necessarily unique. We can easily specify agents’ preferences to generate a second equilibrium in which exactly 50 people vote for each alternative (and so everyone is pivotal).

Suppose we add a voter who prefers a according to both \( u^O \) and \( f \). Now there are 51 individuals who prefer to vote for a if they are pivotal and \( K+1 > 52 \) who prefer to vote for a if they are not pivotal. In this scenario, there is a unique pure strategy Nash equilibrium in which \( K + 1 \) people vote for a.

\(^4\)For this to be an equilibrium, the following inequalities must hold:  
\[ \frac{1}{2} (u_i^O(a) - u_i^O(b)) \geq (f_i(b) - f_i(a)) \]
for each agent \( i \) who votes for a and 
\[ \frac{1}{2} (u_i^O(b) - u_i^O(a)) \geq (f_i(a) - f_i(b)) \]
for each agent \( i \) who votes for b.
Remember that without the additional voter, the remaining 100 voters may split evenly between \(a\) and \(b\). With the additional voter, however, alternative \(a\) wins with \(K + 1 > 52\) votes (possibly unanimously). The knife-edge case where 51 vote for \(a\) and 50 vote for \(b\) is no longer an equilibrium. Why? No one who votes for \(b\) is pivotal, and \(K + 1 > 52\) individuals prefer to vote for \(a\) given they are not pivotal. Thus, at least \(K - 50 > 1\) agents have an incentive to deviate, so this cannot be a Nash equilibrim.

Examples 1 and 2 demonstrate the important effect of \(f\) on the set of Nash equilibria in voting games. Even though \(f\) may have an arbitrarily small effect on each individual’s overall utility function, its existence affects voter behavior in a surprisingly powerful manner. This is because the effect of an individual’s vote on his payoff is highly contingent on the voting profile of other agents. In particular, whether or not voter \(i\) is pivotal has profound effects on his voting behavior. While, in a loose sense, individuals’ preferences over outcomes may be much stronger, the outcome-independent component of preferences ultimately dictates how non-pivotal voters behave.

3 Incomplete Information

Our main result demonstrates that individual voter behavior is highly contingent on the voting profile of other agents. In particular, the way each agent votes hinges on whether he is pivotal. This invites the question: when are voters pivotal? Our intuition is that voters are unlikely to be pivotal when the total number of voters is large.

For example, individual voters are very unlikely to be pivotal in large elections, such as presidential elections in the United States. Indeed, voters (correctly) rarely see themselves as pivotal in these elections. We often hear voters say, “My vote won’t make a difference,” especially in states that historically lean strongly toward one party. Even “swing states” are often decided by thousands, tens of thousands, or even hundreds of thousands of votes. In these cases, voters are very unlikely to be pivotal, and their realization of this fact may affect
their voting behavior.

In this section, we (1) formalize the notion that voters are unlikely to be pivotal when the total number of voters is large and (2) consider implications for voting games with incomplete information.

**Proposition 2.** Suppose outcomes are decided by plurality rule and that given any voter $i$ and any alternative $a_k$, the probability voter $i$ votes for alternative $a_k$ is at most $\bar{p}_k < 1$. Then the probability that any voter is pivotal tends to zero as the number of voters tends to infinity.

*Proof.* See Appendix.

The proof of Proposition 2 requires a lemma about the distribution of the sum of $n$ independent Bernoulli random variables. We state and prove this lemma in the appendix. To prove Proposition 2, we bound the probability that voter $i$ is pivotal by considering the conditions necessary (but not sufficient) for voter $i$ to be pivotal between any two outcomes. We show that the probability that these conditions hold tends to zero as $N$ tends to infinity. See the appendix for the full proof.

Proposition 2 allows us to comment on the expected outcome of voting games with incomplete information when the number of voters is large.

Let us relax Assumption 1 and consider a standard game of incomplete information in which $N \geq 2$ voters decide between $M \geq 2$ alternatives. In the first stage, Nature assigns each voter a pair of functions, $u_i^O$ and $f_i$. These types are drawn independently from some probability distribution. Voters’ types are their private information. In the second stage, players cast their votes simultaneously. Assume outcomes are decided by plurality rule and that ties are broken randomly and with equal probability.\(^5\)

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\(^5\)Our result also holds for majority rule or for any supermajority rule.
Our result requires the following conditions:

**Condition 1.** For each alternative \( k \), each voter has probability at least \( p_k > 0 \) of being assigned preferences such that voting for \( k \) is a strictly dominated strategy.

**Condition 2.** For any distinct alternatives \( a_k \) and \( a_j \), there exist \( \epsilon_{k,j} > 0 \) and \( R_{k,j} < \infty \) such that for each voter \( i \), \( |f_i(a_k) - f_i(a_j)| \geq \epsilon_{k,j} \) and \( |u^O_i(a_k) - u^O_i(a_j)| \leq R_{k,j} \).

These conditions are not as restrictive as they might initially appear. Condition 1 requires that for each alternative there is *some* probability, perhaps arbitrarily small, that voting for that alternative is a strictly dominated strategy. Condition 2 is only a weak extension of Assumption 1, which requires that the preferences induced by \( f_i \) and \( u^O \) are strict. Assumption 1 guarantees that both \( f_i(a_k) - f_i(a_j) \) and \( u^O_i(a_k) - u^O_i(a_j) \) are nonzero, and, in particular, that \( |f_i(a_k) - f_i(a_j)| \geq \epsilon_{k,j} \) holds for any finite set of voters. Condition 2, then, simply guarantees that as \( N \to \infty \), \( (f_N(a_k) - f_N(a_j)) \to 0 \) and \( (u^O_N(a_k) - u^O_N(a_j)) \to \infty \). (In fact, it guarantees this for any subsequence.)

**Proposition 3.** Given a sufficiently large number of voters, there is a unique pure strategy Bayesian Nash equilibrium in which each player \( i \) votes according to \( f_i \).

**Proof.** Proposition 2 allows us to establish this result without studying the properties of the set of Bayesian Nash equilibria in depth. By Condition 1, we have that each agent votes for each alternative \( a_k \) with probability at most \( 1 - p_k < 1 \). Applying Proposition 2, we see that in any pure strategy Bayesian Nash equilibrium, the probability that any voter is pivotal becomes arbitrarily small as the number of voters grows. Let \( p_{j,k} \) be the probability that voter \( i \) is pivotal between outcomes \( a_j \) and \( a_k \). Then the expected benefit to voter \( i \) from voting for alternative \( a_j \) over \( a_k \) can be expressed as

\[
\mathbb{E}[u_i(a_j) - u_i(a_k)] = p_{j,k} (u^O_i(a_j) - u^O_i(a_k)) + f_i(a_j) - f_i(a_k).
\]

By Condition 2, \( |u^O_i(a_j) - u^O_i(a_k)| \leq R_{k,j} \) and \( |f_i(a_j) - f_i(a_k)| \geq \epsilon_{k,j} \), so that if \( p_{j,k} < \frac{\epsilon_{k,j}}{R_{k,j}} \), the expected benefit to voter \( i \) is higher from voting for \( a_j \) if and only if \( f_i(a_j) > f_i(a_k) \). As
$N \to \infty$, $p_{j,k} \to 0$, so this must be the case for sufficiently large $N$. Since $i, a_j$, and $a_k$ are arbitrary, this establishes that there is a unique pure strategy Bayesian Nash equilibrium in which each player $i$ votes for the alternative that maximizes $f_i$. \hfill \square

Proposition 3 suggests that for large enough $N$, rational players do not vote to affect outcomes. Instead, they cast votes according to the outcome-independent component of their preferences (as reflected in $f$).\(^6\) Again, $f$ may represent bribes, threats, desire for social standing, payoffs associated with conformity, or a host of other factors unrelated to voters’ preferences over outcomes.

Some voters denounce other voters who “waste” their vote on obscure third party candidates, when they could “make a difference” by voting for one of two major candidates. Our result suggests the logic works in reverse: the probability that these voters affect the outcome (that is, “make a difference”) is extremely small, so they simply vote according to $f$.

**Another Source of Uncertainty**

Voter uncertainty may not be due to incomplete information at all, but rather due to the difficulty of coordinating on a particular equilibrium. Consider a scenario similar to Example 2. Suppose there are 101 voters who have identical preferences: each prefers to vote ‘yes’ on some proposition based on the outcome-independent component of his preferences, but each prefers to vote ‘no’ if he is pivotal. Then if any subset of 51 voters vote ‘no’ while the others vote ‘yes,’ this forms a pure strategy Nash equilibrium. How did the voters decide which 51 voted ‘no’? In many real world settings, such coordination is difficult, especially if the number of voters is large. This may lead to uncertainty about how other players will vote even with complete information.

\(^6\)Of course, it may be the case that the outcome-dependent and outcome-independent component of their preferences agree.
4 Why Vote?

Let us relax Assumption 2 and suppose voting is optional. Our model of voting offers an interesting perspective on the classic question: why do people vote? If voters are unlikely to alter the outcome of a voting game, our model suggests $f$ will dictate whether or not they vote as well as which alternative they vote for. In particular, for large $N$, we take the Bayesian Nash equilibrium in which each agent $i$ votes according to $f_i$ as the best prediction for the voting game. That is, in large elections, rational voters do not cast votes in order to affect the outcome. Rather, they vote to make a statement, to impress friends, or to spite enemies, or perhaps they vote out of tradition, out of respect, or out of fear. In short, people vote because of $f$.

If there is a cost to voting, then the decision to vote hinges on the magnitude of $f$. Suppose voter $i$ faces some cost to voting, $c_i$. For large enough $N$, the outcome does not depend on voter $i$’s individual vote. However, assume voter $i$ still benefits (or suffers) from the outcome whether or not he actually votes. In this case, voter $i$ solves

$$
\max_{v_i \in V, \lambda \in \{0, 1\}} u_i(v_i) = u_i^O(O(\bar{v})) + \lambda(f_i(v_i) - c_i),
$$

where $\lambda = 1$ if voter $i$ votes and $\lambda = 0$ otherwise. Because $u_i^O(O(\bar{v}))$ does not depend on $v_i$, voter $i$ votes if and only if $f_i(v_i) - c_i > 0$, where $v_i$ maximizes $f_i$.

In Section 2, we described how $f$ can affect the set of Nash equilibria in voting games, even if $f$ is arbitrarily small. In understanding why people vote, however, the size of $f$ is critical. In voting games with sufficiently large numbers of voters, the maximized $f_i$ must exceed $c_i$ for each agent $i$ who chooses to vote. That is, non-outcome-based incentives to vote must outweigh the cost to vote. There are many potential costs to voting, including time spent on registration, rearranging work schedules, transportation to and from the polls, and time spent researching issues or candidates. For each of the millions of citizens who participate in non-pivotal elections, the maximized $f_i$ must outweigh these costs. Those
for whom the maximized $f_i$ does not exceed $c_i$ simply stay home. Thus, even though an arbitrarily small $f$ has important effects on the set of Nash equilibria in voting games, it is not necessarily the case that $f$ is negligible. On the contrary, it appears $f$ motivates millions of non-pivotal voters to cast votes in voluntary elections.

5 Public Versus Secret Ballot

Whether voting is public or secret can affect voter behavior and—as a result—voting outcomes. When votes are observed, each agent’s vote may elicit some response. A voter may meet reward or retaliation for his actions, and this response can come from other voters or agents external to the voting game. For example, a congressman may receive favorable treatment for his district in an appropriations bill written by other legislators if he votes to confirm some federal judge. Or, the same congressman may receive campaign donations from an external lobbying firm if he votes to confirm the federal judge. Here, rewards come from agents internal and external to the voting game. The congressman may also face punishments for his vote in addition to rewards.

Our model accommodates the differences between public and secret voting quite well. Since agents’ preferences over outcomes do not depend on whether voting is public or secret, $u^O$ is unaffected. If voting is public, each agent faces a response to his vote, $R_i(v_i)$. His payoff from this response is $g_i(R_i(v_i))$, where $g : R_i \to \mathbb{R}$ is some function with domain $R_i$, the set of all possible responses to votes cast by agent $i$. We write $f_i(v_i) = g_i(R_i(v_i)) + h_i(v_i)$, and we have

$$u_i(v_i) = u^O_i(O(v_{-i}, v_i)) + g_i(R_i(v_i)) + h_i(v_i),$$

where $h_i(v_i)$ is an outcome-independent component of agent $i$’s payoff unrelated to responses to his vote. Under secret ballot, $g(\cdot)$ still exists but there will be no response because $v_i$ is unobserved.

In Sections 2 and 3, we described how even arbitrarily small $f$ has important effects on
the set of equilibria in voting games. If \( g \) is a relatively large component of \( f \), then we expect whether voting is public or secret to also have important effects on the outcomes of voting games. It may be the case that

\[
f_i^{SECRET}(v) = g_i(\cdot) + h_i(v) > g_i(\cdot) + h_i(v') = f_i^{SECRET}(v')
\]

but that

\[
f_i^{PUBLIC}(v) = g_i(R_i(v)) + h_i(v) < g_i(R_i(v')) + h_i(v') = f_i^{PUBLIC}(v')
\]

for some alternatives \( v \) and \( v' \). If these two are the only alternatives, then in any Nash equilibrium, each non-pivotal voter will vote for \( v \) under secret voting but \( v' \) under public voting. Thus, the Nash equilibria under public voting are almost entirely different from the equilibria under secret voting.\(^7\)

6 Applications

Bribes and Threats

Consider an external player who attempts to influence the outcome of a voting game by either bribing or threatening voters. Suppose voting is public so that the external player can condition his bribes or threats on each agent’s vote. Then by offering arbitrarily small bribes to these voters (and therefore affecting \( f \)), the external player can induce an equilibrium in which some arbitrary outcome \( a \) wins, even if all voters strictly prefer an alternative outcome.

As in Dal Bó (2007), suppose this external player offers voters a bribe which is contingent on whether or not that particular voter is pivotal. That is, this external player offers to pay some voter \( x_i^p \) if \( i \) is pivotal and \( x_i^n \) if \( i \) is not pivotal. By offering a large bribe to pivotal

\(^7\)The only equilibria that may not change are the ones where there are an even number of voters and exactly half of them vote for each alternative (that is, equilibria in which every voter is pivotal).
voters and a smaller one to non-pivotal voters, the player can influence the voting game so that there is a unique equilibrium in which a particular alternative wins by a wide margin. Because the alternative wins non-pivotally, the external agent never needs to pay the larger bribe, $x_p$.

**Committees**

Consider the difference between public and secret committee voting where the committee chair exerts some influence over the voters. This may not be due to explicit bribes or threats, and the committe chair may not necessarily use his influence intentionally. However, if his influence outweighs the effect of $u_i^O$ for all $i$, then there is an equilibrium in which every member votes for the committee chair’s most preferred alternative. Committee members may also exert some influence on each other, perhaps asymmetrically, and therefore affect voting outcomes.

**Legislatures**

Consider a politician voting between two alternatives, $a$ and $b$, where outcome $b$ is more politically feasible. Even if the politician is willing to sacrifice his chances of being re-elected in order to ensure that outcome $a$ is selected, he may vote for outcome $b$ if he does not believe that his vote will swing the outcome toward $a$. As this may be true for many legislators, it could be the case that outcome $b$ is selected in a landslide victory even though a majority of the legislature would prefer to vote for $a$ if they believed their votes to be pivotal.

**Altruism**

Our model may seem like a somewhat depressing interpretation of reality. In pondering $f$, we often think of bribes, threats, corrupt influence, or self-serving motives. Some would rather believe people vote for policies and candidates they think are best for society. Our model accounts for these motives rather well. Suppose an individual gets personal satisfaction from
voting for a socially redistributive policy, even though she strictly prefers the status quo. If \( \Delta f_i \) exceeds \( \Delta u_i^O \), or if he thinks his vote is unlikely to be pivotal, he will vote altruistically.

**Stubborn Voters**

Consider “stubborn” voters, who derive additional personal satisfaction (through \( f \)) by voting according to their outcome preferences (according to \( u^O \)). Then even if \( u_i^O(a) > u_i^O(b) > u_i^O(c) \) and voter \( i \) could be pivotal between \( b \) and \( c \), he may still vote for \( a \) (even if his vote results in \( c \), his least favorite outcome) because of the strength of \( f_i \). When these stubborn voters are not pivotal, they will still, of course, vote according to \( f_i \). In cases of external influence, it will be harder to affect the way these “stubborn” individuals vote.

**Mandates**

In large elections, candidates who receive a sizable majority of the vote often claim a mandate, or say “the electorate has spoken.” Is this true? According to our model, perhaps not. Just because a candidate receives an overwhelming number of votes does not necessarily mean an overwhelming number of voters prefer this candidate. Because no voter is pivotal, all voters are simply responding to outcome-independent incentives.

**7 Conclusion**

Motivated by our intuition that voters respond to incentives unrelated to the outcome of a vote, we develop a game-theoretic model of voting in which we decompose voter preferences into outcome-dependent and outcome-independent components. Outcome-independent components of preferences can dramatically affect the set of Nash equilibria because they dictate the behavior of non-pivotal voters. Why? Voters who are not pivotal cannot, by definition, vote to influence the outcome of the vote. They must therefore vote to maximize their outcome-independent payoff. The only equilibria in which at least one voter does not vote
in accordance with the outcome-independent component of his preferences are equilibria in which there exist pivotal voters.

We then construct a game of incomplete information in which Nature first assigns to each voter a type, specifying the outcome-dependent and outcome-independent components of his preferences. Voters’ types are their private information, and voters cast ballots simultaneously. Within this framework, we formalize the notion that voters are unlikely to be pivotal when the total number of voters is large. In particular, we prove that the probability any voter is pivotal tends to zero as the number of voters tends to infinity. With this result, we can show that given a sufficiently large number of voters, there exists a unique pure strategy Bayesian Nash equilibrium in which each player votes according to the outcome-independent component of his preferences. In voting games with incomplete information and many voters, outcome-independent payoffs truly reign supreme.

Our model offers perspective on the timeless question: why vote? Suppose voting is optional and participation is costly. Further suppose the number of voters is large and, as such, the probability of a pivotal vote is small. Then for the individuals who participate, outcome-independent payoffs must exceed the cost of voting. Thus, even though arbitrarily small outcome-independent payoffs can dramatically affect the set of Nash equilibria in voting games, it is not necessarily the case that these payoffs are negligible. After all, millions of non-pivotal voters bear very real costs to cast ballots in national elections.

Outcome-independent components of preferences are also relevant to the topic of public and secret voting. We expect outcome-independent payoffs to exert greater influence under public voting as they may include societal responses to observed votes. Consider, for example, committee members voting under the watchful eyes of a committee chair or legislators voting under the watchful eyes of their constituents. Can we be sure these actors vote for the outcome they most prefer? Or might they respond to outcome-independent payoffs in the form of bribes, threats, political pressure, or otherwise personal gain?
8 Appendix

In order to prove the lemma stated in this appendix, we will use an adapted version of the Berry-Esseen theorem as developed by Batirov, Manevich, and Nagaev (1977). The Berry-Esseen theorem provides a uniform bound on the rate of convergence of independent and identically distributed (i.i.d.) random variables to the standard normal distribution. Batirov, Manevich, and Nagaev consider a random sum of random variables that are not necessarily i.i.d. Below, we provide a simpler statement of the theorem that assumes the number of terms in the random sum is some fixed integer $N$. For a more straightforward statement of Batirov, Manevich, and Nagaev’s result, see Chaidee and Tuntapthai (2009).

**Theorem** (adapted from Chaidee and Tuntapthai (2009))

Suppose $X_1, X_2, \ldots$ are independent but not necessarily identically distributed random variables with $E[X_i] = 0$, $E[X_i^2] = \sigma_i^2$, and $E[|X_i|^3] = \gamma_i < \infty$. Define

$$S_N^2 = \sum_{i=1}^N \sigma_i^2, \quad \beta_N = \sum_{i=1}^N \gamma_i, \quad \text{and} \quad Y_N = \frac{X_1 + \ldots + X_N}{S_N}$$

Then there exists a constant $C$ such that

$$\sup_{x \in \mathbb{R}} |P\{Y_N \leq x\} - \Phi(x)| \leq C \left( \frac{\beta_N}{(S_N^2)^{3/2}} \right).$$

By this result, if $\frac{\beta_N}{(S_N^2)^{3/2}} \to 0$, then $Y_N$ not only converges in distribution to the standard normal distribution, it converges uniformly.

**Lemma.** Let $Z_N = \sum_{i=1}^N z_i$ be the sum of $N$ independent random variables where $z_i \sim Bernoulli(p_i)$ for $p_i \leq \bar{p} < 1$. Then for any fixed $M \geq 2$,

$$\lim_{N \to \infty} \left( \max_{k \geq \lceil \frac{N}{M} \rceil} P\{Z_N = k\} \right) = 0$$
Proof. Let \( \{z_n\}_{n \in \mathbb{N}} \) be any sequence of independent Bernoulli random variables. That is \( \mathbb{P}\{z_i = 1\} = p_i \leq \bar{p} < 1 \) and \( \mathbb{P}\{z_i = 0\} = 1 - p_i \). Define \( X_i = z_i - \mathbb{E}[z_i] \) so that \( \mathbb{E}[X_i] = 0 \). Define \( \sigma_i^2, S_N^2, \beta_N \), and \( Y_N \) as in the theorem above. Let \( S_N = \sqrt{S_N^2} \).

**Case 1:** Suppose that \( \lim_{N \to \infty} \frac{\sum_{i=1}^{N} \sigma_i^2}{N} \neq 0 \).

We break down the proof into a series of steps.

**Step 1:** \( \frac{\beta_N}{(S_N^2)^3} \to 0 \) and \( S_N \to \infty \).

By definition, \( \beta_N = \sum_{i=1}^{N} \mathbb{E}[|X_i|^3] \leq N \), where the last inequality follows because \( X_i \) only takes on values of 0 and 1. This implies:

\[
\frac{\beta_N}{(S_N^2)^3} \leq \frac{N}{\left( \sum_{i=1}^{N} \sigma_i^2 \right)^3} \leq \frac{1}{\left( \frac{N^2}{N^3} \right) \left( \sum_{i=1}^{N} \sigma_i^2 \right)^3} = \frac{1}{N^2 \left( \frac{\sum_{i=1}^{N} \sigma_i^2}{N} \right)^3} \to 0
\]

where the last step follows because \( \left( \frac{\sum_{i=1}^{N} \sigma_i^2}{N} \right) \to 0 \).

To see why \( S_N \to \infty \), note that because \( \lim_{N \to \infty} \sum_{i=1}^{N} \sigma_i^2 \neq 0 \), it must be that for some \( i, \sigma_i \neq 0 \), which implies \( p_i > 0 \). Therefore, we know \( \beta_N \not\to 0 \) because \( \beta_N \geq \mathbb{E}[|X_i|^3] = p_i^3 \). Since \( \beta_N \not\to 0 \), \( \frac{\beta_N}{(S_N^2)^3} = \frac{\beta_N}{(S_N)^6} \to 0 \) implies \( S_N \to \infty \).
**Step 2:** We can rewrite $\mathbb{P}\{Z_N = k\}$ in the following way:

$$
\mathbb{P}\{Z_N = k\} = \mathbb{P}\{Z_N \in (k - 1, k + 1)\}
$$

$$
= \mathbb{P}\{z_1 + \cdots + z_N \in (k - 1, k + 1)\}
$$

$$
= \mathbb{P}\left\{ \frac{(z_1 - \mu_1) + \cdots + (z_N - \mu_N)}{S_N} \in \left( \frac{k - 1 - \mu_1 - \cdots - \mu_N}{S_N}, \frac{k + 1 - \mu_1 - \cdots - \mu_N}{S_N} \right) \right\}
$$

$$
= \mathbb{P}\left\{ \frac{X_1 + \cdots + X_N}{S_N} \in \left( \frac{k - 1 - \mu_1 - \cdots - \mu_N}{S_N}, \frac{k + 1 - \mu_1 - \cdots - \mu_N}{S_N} \right) \right\}
$$

$$
= \mathbb{P}\left\{ Y_N \in \left( \frac{k - 1 - \mu_1 - \cdots - \mu_N}{S_N}, \frac{k + 1 - \mu_1 - \cdots - \mu_N}{S_N} \right) \right\},
$$

where $Y_N = \frac{X_1 + \cdots + X_N}{S_N}$ as in the theorem above.

**Step 3:** The necessary conditions for the theorem stated above are satisfied. This follows because $\frac{\beta_N}{(s_N^2)} \to 0$ and $\mathbb{E} |X_i|^\beta$ is obviously finite for all $i$.

**Step 4:** For sufficiently large $N$, the probability $Y_N$ is within any interval is arbitrarily close to the probability a variable that follows the standard normal distribution falls within the same interval.

Fix $\epsilon > 0$. Pick $N_1 \in \mathbb{N}$ such that $N > N_1$ implies $C \left( \frac{\beta_N}{(s_N^2)} \right) < \frac{\epsilon}{2}$, where $C$ is the
constant from the statement of the theorem. Then,

$$\sup_{x_1, x_2 \in \mathbb{R}} |(\mathbb{P}\{Y_N \leq x_1\} - \mathbb{P}\{Y_N \leq x_2\}) - (\Phi(x_1) - \Phi(x_2))|$$

$$\leq \sup_{x_1 \in \mathbb{R}} |\mathbb{P}\{Y_N \leq x_1\} - \Phi(x_1)| + \sup_{x_2 \in \mathbb{R}} |\mathbb{P}\{Y_N \leq x_2\} - \Phi(x_2)|$$

$$\leq 2C \left( \frac{\beta_N}{(S_N^2)^2} \right)$$

$$< \epsilon$$

**Step 5**: \(\sup_{k \in \mathbb{R}} \left( \Phi \left( \frac{k+1-\mu_1-...-\mu_N}{S_N} \right) - \Phi \left( \frac{k-1-\mu_1-...-\mu_N}{S_N} \right) \right) \to 0.\)

First, recall that \(S_N \to \infty\). Therefore, the interval \(\left( \frac{k-1-\mu_1-...-\mu_N}{S_N}, \frac{k+1-\mu_1-...-\mu_N}{S_N} \right)\), which has area \(\frac{2}{S_N}\), converges to a single point as \(N \to \infty\). The claim follows by the uniform continuity of the cumulative distribution function of the standard normal distribution.

**Step 6**: Combining steps 2, 4, and 5 proves that the lemma holds in this case.

Fix any \(\epsilon > 0\). By Step 4, we can pick \(N_1\) such that for any \(N > N_1\)

$$\sup_{x_1, x_2 \in \mathbb{R}} |(\mathbb{P}\{Y_N \leq x_1\} - \mathbb{P}\{Y_N \leq x_2\}) - (\Phi(x_1) - \Phi(x_2))| < \frac{\epsilon}{2}$$

By Step 5, there exists \(N_2\) such that \(N > N_2\) implies

$$\sup_{k \in \mathbb{R}} \left( \Phi \left( \frac{k+1-\mu_1-...-\mu_N}{S_N} \right) - \Phi \left( \frac{k-1-\mu_1-...-\mu_N}{S_N} \right) \right) < \frac{\epsilon}{2}$$
Pick $N_3 = \max\{N_1, N_2\}$. Then, applying step 2, for any $N > N_3$,

$$
\max_{k \geq \left\lfloor \frac{N_3 - 1}{M} \right\rfloor} \mathbb{P} \{ Z_N = k \} = \max_{k \geq \left\lfloor \frac{N_3 - 1}{M} \right\rfloor} \left( \mathbb{P} \left\{ Y_N \leq \frac{k + 1 - \mu_1 - \ldots - \mu_N}{S_N} \right\} - \mathbb{P} \left\{ Y_N \leq \frac{k - 1 - \mu_1 - \ldots - \mu_N}{S_N} \right\} \right)
$$

$$
< \max_{k \in \left\{ \left\lfloor \frac{N_3 - 1}{M} \right\rfloor, \ldots, N \right\}} \left( \Phi \left( \frac{k + 1 - \mu_1 - \ldots - \mu_N}{S_N} \right) - \Phi \left( \frac{k - 1 - \mu_1 - \ldots - \mu_N}{S_N} \right) \right) + \frac{\epsilon}{2}
$$

$$
< \epsilon
$$

**Case 2:** Suppose $\lim_{N \to \infty} \frac{\sum_{i=1}^{N} \sigma_i^2}{N} = 0$.

We know that $\frac{\sum_{i=1}^{N} \sigma_i^2}{N} = \frac{\sum_{i=1}^{N} (1-p_i) p_i}{N}$. Because $p_i \leq \bar{p} < 1$ for each $i$, it must be that $\lim_{N \to \infty} \frac{\sum_{i=1}^{N} p_i}{N} = 0$. But if the average parameter tends to zero, then $\mathbb{P} \{ Z_N \geq \left\lfloor \frac{N - 1}{M} \right\rfloor \} \leq \mathbb{P} \left\{ \frac{z_1 + \ldots + z_N}{N} \geq \frac{1}{M} \right\} \to 0$.  

**Proof of Proposition 2.** Define $E_{a,b}^i$ to be the event that voter $i$ is pivotal between $a$ and $b$. For any voter $j$, denote $j$’s probability of voting for alternative $k \in \{a_1, \ldots, a_M\}$ by $p_j^k$.

We look at subsets of voters $A$, $B$, $Z$ and $Y$ as follows:

$$
A_N := \{ j \in \{1, \ldots, i - 1, i + 1, \ldots, N \} \mid p_j^a > 0 \text{ and } p_j^b = 0 \}
$$

$$
B_N := \{ j \in \{1, \ldots, i - 1, i + 1, \ldots, N \} \mid p_j^a = 0 \text{ and } p_j^b > 0 \}
$$

$$
Z_N := \{ j \in \{1, \ldots, i - 1, i + 1, \ldots, N \} \mid p_j^a = 0 \text{ and } p_j^b = 0 \}
$$

$$
Y_N := \{ j \in \{1, \ldots, i - 1, i + 1, \ldots, N \} \mid p_j^a > 0 \text{ and } p_j^b > 0 \}
$$

Here, $A$ is the set of voters other than $i$ who vote for $a$ with positive probability but who never vote for $b$, and $B$ is defined analogously. The voters in $Z$ never vote for $a$ or $b$, while those in $Y$ vote for each $a$ and $b$ with positive probability.
Define the related random variables $X^N_a$, $X^N_b$, $Y^N_a$, and $Y^N_b$ as follows:

\[
X^N_a := \# \{ j \in A_N \mid v_j = a \} \\
X^N_b := \# \{ j \in B_N \mid v_j = b \} \\
Y^N_a := \# \{ j \in Y_N \mid v_j = a \} \\
Y^N_b := \# \{ j \in Y_N \mid v_j = b \}
\]

In order for voter $i$ to be pivotal between $a$ and $b$, it must be that (1) the number of votes for $a$ is within one vote of the number of votes for $b$ and (2) the number of votes for $b$ (or $a$) is at least $\lceil \frac{N-1}{M} \rceil$. The second requirement follows because otherwise alternative $b$ could never win, so $i$ could not be pivotal.

These conditions allow us to bound the probability that voter $i$ is pivotal between $a$ and $b$.

\[
\mathbb{P} \{ E^i_{a,b} \} \leq \mathbb{P} \left\{ X^N_a + Y^N_a = X^N_b + Y^N_b \geq \left\lceil \frac{N-1}{M} \right\rceil \right\} + \mathbb{P} \left\{ X^N_a + Y^N_a = X^N_b + Y^N_b + 1 \geq \left\lceil \frac{N-1}{M} \right\rceil \right\}
\]

Which implies:

\[
\mathbb{P} \{ E^i_{a,b} \} \leq \sum_{k \geq \left\lceil \frac{N-1}{M} \right\rceil} \mathbb{P} \{ X^N_a + Y^N_a = k \} \mathbb{P} \{ X^N_b + Y^N_b \in \{ k, k+1 \} \mid X^N_a + Y^N_a = k \}
\]

\[
\leq \max_{k \geq \left\lceil \frac{N-1}{M} \right\rceil} \mathbb{P} \{ X^N_a + Y^N_a = k \} \sum_{k \geq \left\lceil \frac{N-1}{M} \right\rceil} \mathbb{P} \{ X^N_b + Y^N_b \in \{ k, k+1 \} \mid X^N_a + Y^N_a = k \}
\]

\[
\leq 2 \times \max_{k \geq \left\lceil \frac{N-1}{M} \right\rceil} \mathbb{P} \{ X^N_a + Y^N_a = k \}
\]

Now, note that $X^N_a + Y^N_a$ is just a sum of independent Bernoulli random variables and that, by assumption, their parameters are bounded above by $\bar{p}_a < 1$. Furthermore, if $X^N_a + Y^N_a$ is the sum of fewer than $\left\lceil \frac{N-1}{M} \right\rceil$ Bernoulli random variables, $\max_{k \geq \left\lceil \frac{N-1}{M} \right\rceil} \mathbb{P} \{ X^N_a + Y^N_a = k \} = 0.$
If, on the other hand, \(X_a^N + Y_a^N\) is the sum of at least \(\left\lfloor \frac{N-1}{M} \right\rfloor\) Bernoulli random variables, picking arbitrarily large \(N\) can guarantee that \(X_a^N + Y_a^N\) is the sum of an arbitrarily large number of independent Bernoulli random variables. Therefore, by the lemma, we have that 
\[
\max_{k \geq \left\lfloor \frac{N-1}{M} \right\rfloor} \mathbb{P}\{X_a^N + Y_a^N = k\} \to 0.
\]
Then, by the inequalities above,
\[
\mathbb{P}\{E_{a,b}^i\} \leq 2 \times \max_{k \geq \left\lfloor \frac{N-1}{M} \right\rfloor} \mathbb{P}\{X_a^N + Y_a^N = k\} \to 0
\]
Because the set of alternatives is finite, making a finite number of pairwise comparisons establishes that the probability voter \(i\) is pivotal between any alternatives tends to zero as the number of voters tends to infinity. Since the selection of voter \(i\) is arbitrary, this result holds for every voter. \(\square\)

References


