University of Michigan Physics Department
Graduate Qualifying Examination

Part I: Classical Physics
Saturday, January 12, 2013 9 am – 2 pm

This is a closed book exam, but a number of useful quantities and formulas are provided in the front of the exam. **(Note that this list is more extensive than in past years.)** If you need to make an assumption or estimate, indicate it clearly. Show your work in an organized manner to receive partial credit for it. Answer the questions directly in this exam booklet. If you need more space than there is under the problem, continue on the back of the page or on additional blank pages that the proctor will provide. Please indicate if you continue your answer on another page. If you use additional blank pages, state clearly at the top of each page your exam number, found at the upper right of this page—but not your name—the question number, and “page x of y” (if there is more than one page per question).

You must answer the first 8 required questions and 2 of the 4 optional questions. Indicate which of the latter you wish us to grade (e.g. by circling the question number). We will only grade the indicated optional questions. Good luck!!

**Some integrals and series expansions**

\[
\int_{-\infty}^{\infty} \exp(-\alpha x^2) \, dx = \sqrt{\frac{\pi}{\alpha}}
\]

\[
\int_{-\infty}^{\infty} x^2 \exp(-\alpha x^2) \, dx = \frac{1}{2} \sqrt{\frac{\pi}{\alpha^3}}
\]

\[
\exp(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots
\]

\[
\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots
\]

\[
\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots
\]

\[
\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots
\]

\[
(1 + x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2} x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} x^3 + \cdots
\]
**Some Fundamental Constants**

speed of light \( c = 2.998 \times 10^8 \) m/s  
proton charge \( e = 1.602 \times 10^{-19} \) C  
Planck’s constant \( \hbar = 6.626 \times 10^{-34} \) J·s = 4.136×10\(^{-15}\) eV·s  
Rydberg constant \( R_n = 1.097 \times 10^7 \) m\(^{-1}\)  
Coulomb constant \( k = (4\pi\varepsilon_0)^{-1} = 8.988 \times 10^9 \) N·m\(^2\)/C\(^2\)  
vacuum permeability \( \mu_0 = 4\pi \times 10^{-7} \) T·m/A  
universal gas constant \( R = 8.3 \) J/K·mol  
Avogadro’s number \( N_A = 6.02 \times 10^{23} \) mol\(^{-1}\)  
Boltzmann’s constant \( k_B = R/N_A = 1.38 \times 10^{-23} \) J/K = 8.617×10\(^{-5}\) eV/K  
Stefan-Boltzmann constant \( \sigma = 5.67 \times 10^{-8} \) W/m\(^2\)K\(^4\)  
radius of the sun \( R_{\text{sun}} = 6.96 \times 10^8 \) m  
radius of the earth \( R_{\text{earth}} = 6.37 \times 10^6 \) m  
radius of the moon \( R_{\text{moon}} = 1.74 \times 10^6 \) m  
gravitational constant \( G = 6.67 \times 10^{-11} \) m\(^3\)/(kg·s\(^2\))
1. (Mechanics) Consider a gravitational potential for an extended distribution of mass (like the dark matter halo of a galaxy) and assume that it has a simple logarithmic form:

\[ \Phi(x, y, z) = \Phi_0 \ln \left( \frac{L^2 + x^2 + y^2 + z^2}{c^2} \right), \]

where the lengths are given in dimensionless units and where \( L, c, \) and \( \Phi_0 \) are constants. For this potential, find the corresponding mass density distribution \( \rho(x, y, z) \). Given your result for this distribution, find a constraint on the constants \((L, c)\) that ensures that the mass density is everywhere positive.
2. **(Mechanics)** A point mass \( m \) glides on a cycloid without friction. The cycloid can be described in parametric form as \( x = R (\varphi - \sin \varphi) \) and \( y = R (1 + \cos \varphi) \), where the parameter \( \varphi > 0 \). The system sits in a uniform gravitational field \((g)\) which is along the \(-y\) direction.

(1) Write down the Lagrangian in terms of \( \varphi \) and \( \dot{\varphi} \) (where \( \varphi \) gives the location of the particle on the cycloid).

(2) Rewrite the Lagrangian in terms of \( u = \cos \frac{\varphi}{2} \) [Hint: \( \cos^2 \frac{\varphi}{2} = \frac{1+\cos \varphi}{2} \) and \( \sin^2 \frac{\varphi}{2} = \frac{1-\cos \varphi}{2} \)].

(3) Find the equation of motion for \( u \).

(4) The equation of motion found in part (3) describes an oscillation. Find the periodicity of this oscillation.
3. (Mechanics) A spinning flywheel with moment of inertia \( I_0 \) about its axis is concealed inside a black box of side \( L \), as shown. The flywheel’s rotation axis goes through the centers of mass of the flywheel and of the box. The box and flywheel together have total mass \( M \) and are supported on a table by supports at A and B (whose size is negligible compared to \( L \)). Friction at the flywheel bearings causes its angular velocity to decrease at a constant rate

\[
\frac{d\omega}{dt} = -\gamma, \\
\]

where \( \gamma > 0 \).

a) What are the normal forces at the supports A and B?

b) If \( \gamma \) is large enough, the box will tip up and start to tumble along the table. For flywheel rotation in the direction shown in the diagram, which of the supports will lift up first? **Justify your answer fully** (no credit will be given for answers that look like random guesses).

c) How large must \( \gamma \) be for this to happen?
4. \textbf{(E&M)} A coaxial transmission line has an inner radius \( a \), an outer radius \( b \), and a (dimensionless) relative dielectric constant \( \varepsilon_r \). Answer the following:

a. What is the capacitance per unit length (in SI units)? In doing the calculation, assume the inner conductor has charge \(+Q\), and the outer conductor has charge \(-Q\).

b. What is the characteristic impedance for a Teflon cable \((\varepsilon_r = 2.1)\) where the inner radius \( a = 1 \text{ mm} \) and the outer radius \( b = 5 \text{ mm} \)?

(For inductance, use the fact \( L = \frac{\mu \varepsilon}{\varepsilon_f} \), where \( \varepsilon = \varepsilon_f \varepsilon_0 \), and assume \( \mu = \mu_0 \)).

c. What is the magnitude of the reflection coefficient if the coax is terminated with a 50 Ohm load?
5. (E&M) Find the energy required to assemble a uniform, solid sphere of charge with radius $R$ and charge density $\rho$. 
6. (E&M) Two circular wire loops of radius $R$ are oriented parallel to the $x$-$y$ plane, with their centers at $(0,0,h)$ and $(0,0,-h)$, respectively.

(a) Suppose that both loops carry an identical current $I$ counterclockwise (as viewed from above, i.e. from a point with $z > h$). Find the leading nonzero behavior of the magnetic field $B$ along the $z$ axis for $|z| \gg R, h$. (Be sure to give answers for both positive and negative $z$, and remember to express the magnetic field as a vector.)

(b) Now suppose that the current in the two loops has the same magnitude $I$ but goes in opposite directions, with the loop at $z=h$ carrying a counterclockwise current and the loop at $z=-h$ carrying a clockwise current (both as viewed from above). Find the leading nonzero behavior of the magnetic field along the $z$ axis for $|z| \gg R, h$. (Be sure to give answers for both positive and negative $z$, and remember to express the magnetic field as a vector.)
7. (Thermodynamics) Consider a rubber band of length \( L \) and tension \( F \) at a temperature \( T \). (By definition, the tension \( F \) is positive when an outside force is pulling on the rubber band so as to try to lengthen it.)

a) Write down a thermodynamic identity relating changes in the energy \( dE \) of the rubber band to changes in its entropy \( dS \) and its length \( dl \). Explain why you wrote this identity in a particular way.

b) Using the empirical fact that a rubber band heats up when stretched, deduce whether the tension increases or decreases if the rubber band is heated at fixed length \( L \). **Note:** No credit will be given for an answer without detailed justification or with any logical steps missing, as there is a 50% chance of guessing the answer correctly in this part.
8. **(Optics)** Rays from a lens are converging toward a point image $P_1$ located to the right of the lens. What thickness $t$ of glass with index of refraction $1.60$ must be interposed between the lens and $P_1$ for the image to be formed at $P_2$, located $0.3$ [cm] to the right of $P_1$? The locations of the piece of glass and of points $P_1$ and $P_2$ are shown in the figure.
Optional (do 2 of 4 problems)

9. (Mechanics) An infinite chain of point masses, each of mass \( m \) and each hanging from a rigid massless rod of length \( L \), is placed in a uniform gravitational field \( g \) as shown in the diagram. Each pair of adjacent point masses is connected by a massless spring of spring constant \( C \) and of non-stretched length \( d \), which is also the equilibrium spacing between the point masses. Throughout this problem, assume that the masses move only in the plane of the diagram.

\[
\begin{align*}
\text{Diagram showing a rigid rod with point masses connected by massless springs.}
\end{align*}
\]

a) Find the Lagrangian of the system for small displacements of each mass of size \( s_i << L \) in the plane of the diagram.

b) Derive the equation of motion of the \( i^{th} \) mass under the same simplifying assumption \( (s_i << L) \).

c) Introduce the linear mass density \( \rho = m/d \) and take the continuum limit of the equation of motion by considering waves with wavelength \( \lambda >> d \).

d) Find the dispersion relation \( \omega = \omega(k) \) for long waves in this system.

e) What is the minimum possible angular frequency for long waves? What normal mode of oscillations does it represent (that is, how do the masses move when they are oscillating at this frequency)?
10. (E&M) A conducting sphere of radius $R$ with net charge $Q$ is placed in a uniform external electric field $\mathbf{E} = E_0 \hat{z}$. (This condition means, of course, uniform in the absence of the conducting sphere, which in turn means uniform far away from the sphere.) The conducting sphere is of course an equipotential, $\Phi(R, \theta) = V_0$. Determine the electric potential $\Phi(r, \theta)$, where $r$ is the distance from the center of the sphere and $\theta$ is as indicated, outside the conducting sphere.
11. (Thermodynamics) Consider a metal with specific heat capacity at constant volume \( c_v = gT \), where \( g = 1 \text{ J}/(\text{K}^2 \text{ kg}) \) is a constant. A 1 kg piece of this metal at a temperature of 5°C is placed in a large container of water, which you may treat as an infinite heat reservoir, at 50°C. The metal and water exchange heat until the system comes to equilibrium. Assume that any changes in volume (of either the metal or the water) are negligible. What is the change in entropy of the combined metal plus water system during this process?
12. **(Optics)** A continuous, collimated Gaussian beam with wavelength \( \lambda = 532 \text{ nm} \), \( w \)-parameter (waist size) of 5 mm, and total power 1 W is to be focused into a small experimental region such that the peak intensity at the center of the focal spot is \( I_0 = 10^7 \frac{W}{cm^2} \). You are supposed to pick a lens for the job. Assume that all light power passes through the aperture of the lens.

(a) What is the central (peak) intensity of the beam before it enters the lens?

(b) What is the beam waist size \( w \) at the focal spot?

(b) What focal length should you pick for the lens? (Assume an ideal lens without aberrations).

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Useful equations: \( r = \text{distance from beam axis}, \ z = \text{distance from focus} \)

- For given \( w \) the intensity follows \( I(r) = I_0 \exp \left( -\frac{2r^2}{w^2} \right) \)

- Near a focal spot, \( w(z) = w_0 \sqrt{1 + \left( \frac{z}{z_R} \right)^2} \), where the Rayleigh length \( z_R = \frac{\pi w_0^2}{\lambda} \) and \( w_0 \) is the \( w \)-parameter (waist size) at the focus.

- Gaussian integrals (see front of the exam).