Name: ______________________ Name-code: ______________________

# University of Michigan Physics Department
# Graduate Qualifying Examination

Part II – Modern Physics
Saturday, January 14, 2006 9:00am-1:00pm

**Please write your name-code on each page of the exam.**

This is a closed book exam – but you may use the “Constants, Conversions, and Formulas” sheet we provide for this exam.

Show your work in an organized manner to receive partial for it. If you need to make an assumption or an estimate, indicate it clearly.

You must answer the first 8 obligatory questions and two of the optional four questions. Indicate which of the latter you wish us to grade (e.g., circle the question number). We will only grade the indicated optional questions.

Please do not write on this page. The hatched column to the right and the total score will be filled in by the graders.

Good Luck!

Total Score:

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| 1 | |
| 2 | |
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## Part A: Obligatory Problems

1. In deriving the Generalized Uncertainty Principle we consider two observables $A$ and $B$ and their corresponding uncertainties given by

$$\begin{aligned} \sigma_A^2 &= \langle (\hat{A} - \langle A \rangle)\Psi | (\hat{A} - \langle A \rangle)\Psi \rangle = \langle f | f \rangle \\ \sigma_B^2 &= \langle (\hat{B} - \langle B \rangle)\Psi | (\hat{B} - \langle B \rangle)\Psi \rangle = \langle g | g \rangle \end{aligned}$$

where

$$|f\rangle \equiv (\hat{A} - \langle A \rangle)|\Psi\rangle, \quad \text{and} \quad |g\rangle \equiv (\hat{B} - \langle B \rangle)|\Psi\rangle.$$

a) Find the necessary and sufficient condition for achieving minimum uncertainty.
b) Show that for a position-momentum uncertainty the criterion implies that the minimum-uncertainty wave packet is a Gaussian.

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2. Consider a particle in the infinite square well potential

$$V = \begin{cases} 0, \text{if } 0 \leq x \leq a \\ \infty, \text{ otherwise} \end{cases}$$

a) Find the corresponding wave functions and the values of the energy.
b) Suppose we put a delta-function bump in the center of the infinite square well. In terms of perturbation theory we are perturbing the problem by a Hamiltonian of the form

$$H' = \alpha\delta(x - a/2).$$

Find the first order correction to the allowed energies.

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3. a) Sketch the energy level structure of a Helium atom for excitation up to $n = 3$. In your diagram, use spectroscopic notation to identify spin $S$, orbital angular momentum $L$ and angular momentum $J$.
   b) Determine the degeneracy of levels with given energy, $S$ and $L$, but unspecified $J$. Account for all magnetic sublevels.
   c) Is there a metastable state? If yes, why?

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4. What are the allowed Bragg scatterings from a 1*d* chain of regularly spaced atoms? Exhibit this as a geometric construction in *k*-space analogous to the Ewald sphere.

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5. In a certain system, identical quantum particles with spin $S = 1/2$ can have energy levels 0 to $E$. Consider the case where two particles are placed into this system.
   a) Determine the partition function.
   b) Calculate the average energy and the heat capacity as a function of temperature.

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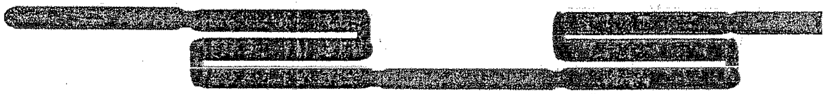
6. Suppose that the polymers that make up the rubber chain are freely jointed macromolecules as shown below. Suppose that each monomer has length $a$, but that the total length of the chain depends on the way it folds. You may assume that at each joint there is an equal probability to take a step to the left or the right. Figure out the entropy associated with a configuration with $N_+$ steps to the right and $N_-$ to the left. Take the limit of large $N$ with $N_+ - N_-$ small. Then show that the force to extend the chain is given by

$$f \propto L$$

where $L = a(N_+ - N_-)$ is the length. Find the constant of proportionality.



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7. A $\pi^0$ of mass $m$ and momentum $p$ decays into two photons. Calculate in the laboratory frame

   a) the energy range of the final-state photons.

   b) the minimum opening angle between the two photons.

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8. Consider a neutron star as some nuclear matter in a gas model. The number of states $N$ in a nuclear volume $V$ is

$$N = \int dn = 2 \times \frac{4\pi \int p^2 dp}{(2\pi\hbar)^3} V, \quad \text{where } V = \frac{4}{3}\pi R^3$$

and $R$ is the radius of the star. The factor of 2 is due to spin degrees of freedom. For a given number $N$ of neutrons, these states are filled to the Fermi momentum $p_F$.
a) Find the average kinetic energy per nucleon, each of mass $m_n$.
b) Find the average gravitational energy per nucleon in this uniform gas. For definiteness, the potential energy between two point masses is

$$-G\frac{m_1 m_2}{r}$$

with $G = 6.71 \times 10^{-39}\, \hbar c(\mathrm{GeV}/c)^{-2}$.
c) Find the equilibrium radius of this configuration for a fixed $N$.
d) What is the radius of the neutron star in this model, if its mass is 1.5 that of the sun, or $N \simeq 1.8 \times 10^{57}$? One number you need is the Compton wave length of the nucleon $\frac{\hbar}{m_n c} \simeq 2.14 \times 10^{-16}\, m$.

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## Part B: Optional Problems

9. A non-relativistic particle with mass $m$ moves in one spatial dimension subject to the "inverted harmonic oscillator potential"

$$V(x) = -\frac{1}{2}\alpha x^2 \; ; \; \alpha > 0.$$

At early times the particle is well-localized at $x \ll 0$, its momentum $p$ is positive, and its energy $E < 0$. Estimate the probability $P(E)$ that this particle is found at $x \gg 0$ at late times.

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10. A tritium atom ($^3H$) in the electronic ground state decays into a Helium ion ($^3He^+$). What is the probability that the ion is in its electronic ground state?

    Note $\int_0^\infty x^n \exp(-px)dx = n!/p^{n+1}$.

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11. In particle physics, the Planck mass $M_P$ characterizes the energy scale at which the gravitational interaction is strong. Thus the gravitational potential energy between two masses ($m_1$ and $m_2$) at a distance $r$ has the form

$$V_0(r) = \frac{\hbar c}{M_P^2} \frac{m_1 m_2}{r}$$

Here $\hbar$ and $c$ are Planck's constant and the speed of light respectively. In a space with $n$ extra spatial dimensions, the energy scale at which gravity is strong is called the string scale ($M_S$). How would the potential energy ($V_n(r)$) depend on $M_S$ and $r$ in this case? If all $n$ extra dimensions are compactified over a small and equal scale $R$, what would be the potential energy from the perspective of our 3-regular spatial dimensions? What is the relationship between $M_S$ and $M_P$?

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12. Excitons can be thought of as bound electron-hole pairs in semiconductors (Wannier excitons). There is another kind, the Frenkel exciton, which occurs in molecular crystals. You may think of it as an excited state of the molecules which hops from site to site – note that the *excitation* hops. (The coupling which leads to excitation transfer is usually electromagnetic.) Consider a wavefunction for a linear chain of molecules:

$$\Psi_n(r_1, \ldots, r_N) = \phi_1(r_1)\phi_2(r_2)\cdots\chi_n(r_n)\phi_{n+1}(r_{n+1})\cdots\phi_N(r_N).$$

In this expression, $\phi_i(r_i)$ is the ground state wavefunction for the electron on molecule $i$ and $\chi_n(r_n)$ is the excited state at site $n$. The hopping matrix element (which you can assume to be non-zero only for nearest neighbors) is called $\Delta$:

$$\Delta = \langle\Psi_n|H|\Psi_{n\pm1}\rangle = \langle\chi_n(r_n)\phi_{n\pm1}(r_{n\pm1})|H|\phi_n(r_n)\chi_{n\pm1}(r_{n\pm1})\rangle$$

The only other matrix element of $H$ which you need consider is $\epsilon = \langle\Psi_n|H|\Psi_n\rangle$, the molecular excited state energy. Write down energy eigenfunctions assuming $H$ acts only in the space of the $\Psi_n$. Sketch the energy band of the excitons. Use periodic boundary conditions. You can assume that $\langle\Psi_n|\Psi_m\rangle = \delta_{m,n}$.