Name: ________________________________

University of Michigan Physics Department Graduate Qualifying Examination

Part II - Modern Physics

Saturday, January 12, 2008   9:00am-1:00pm

This is a closed book exam - but you may use the materials provided at the exam. If you need to make an assumption or estimate, indicate it clearly. Show your work in an organized manner to receive partial credit for it.

You must answer the first 8 obligatory questions and two of the optional four questions. Indicate which of the latter you wish us to grade (e.g., circle the question number). We will only grade the indicated optional questions. Good Luck.

SOME FUNDAMENTAL CONSTANTS IN CONVENIENT UNITS

$$\text{speed of light} \quad c = 2.998 \times 10^{8} \text{ m/s}$$

$$\text{electron charge} \quad e = 1.602 \times 10^{-19} \text{ C}$$

$$\text{Planck's constant} \quad h = 6.626 \times 10^{-34} \text{ J} \cdot \text{s} = 4.136 \times 10^{-15} \text{ eV} \cdot \text{c}$$

$$\hbar = h/2\pi = 1.055 \times 10^{-34} \text{ J} \cdot \text{s} = 0.658 \times 10^{-15} \text{ eV} \cdot \text{s}$$

$$\text{Rydberg constant} \quad R_\infty = 1.097 \times 10^{6} \text{ m}^{-1}$$

$$\text{Coulomb constant} \quad k = (4\pi\epsilon_0)^{-1} = 8.988 \times 10^{9} \text{ N} \cdot \text{m}^2/\text{C}^2$$

$$\text{Universal gas constant} \quad R = 8.3 \text{J/K} \cdot \text{mol}$$

$$\text{Avogadro's number} \quad N_A = 6 \times 10^{23} \text{ mol}^{-1}$$

$$\text{Boltzmann's constant} \quad k_B = R/N_A = 1.38 \times 10^{-23} J/K = 8.617 \times 10^{-5} \text{eV/K}$$

$$\text{Stefan} - \text{Boltzmann constant} \quad \sigma = 5.6703 \times 10^{-8} \text{ W/m}^2\text{K}^4$$

$$\text{radius of the sun} \quad R_{sun} = 6.96 \times 10^{8} \text{ m}$$

$$\text{radius of the moon} \quad R_{moon} = 1.74 \times 10^{6} \text{ m}$$

$$\text{radius of the earth} \quad R_{earth} = 6.37 \times 10^{6} \text{ m}$$

$$G_N = 6.67 \times 10^{-11} m^3/kg/s^2 = 6.71 \times 10^{-39} GeV^{-2}$$

SOME USEFUL CONVERSIONS AND COMBINATIONS

$$\text{fine structure constant} \quad \alpha = ke^2/\hbar c = 1/137$$

$$\text{Bohr magneton} \quad e\hbar/2m_e = 9.27 \times 10^{-24} \text{J/T} = 5.79 \times 10^{-5} \text{eV/T}$$

$$hc = 19.865 \times 10^{-26} \text{ J} \cdot \text{m} = 12.41 \times 10^{3} \text{ eV} \cdot \text{Å} = 1241 \text{ MeV} \cdot \text{fm}$$

$$\hbar c = 3.165 \times 10^{-26} \text{ J} \cdot \text{m} = 1973 \text{ eV} \cdot \text{Å} = 197.3 \text{ MeV} \cdot \text{fm}$$

$$ke^2 = 1.44 \text{ MeV} \cdot \text{fm}$$

$$1\text{Å} = 10^{-10} \text{ m} = 10^{5} \text{ fm} \qquad 1eV = 1.602 \times 10^{-19} \text{ J}$$

SOME USEFUL RELATIONS

$$\frac{C_V}{Nk} = 9\,(T/\theta_D)^3 \int_0^{\theta_D/T} \frac{x^4 e^4 dx}{(e^x - 1)^2} \quad \text{(Debye formula)}$$

$$= 3\left[1 - \frac{1}{20}\left(\frac{\theta_D}{T}\right)^2 + \ldots\right] = \frac{12\pi^4}{5}\left(\frac{T}{\theta_D}\right)^3 (1 + \ldots)$$

$$\frac{U}{N} = \frac{3}{5}E_F\left[1 + \frac{5\pi^2}{12}\left(\frac{kT}{E_F}\right)^2 - \frac{\pi^4}{16}\left(\frac{kT}{E_F}\right)^4 + \ldots\right] \quad \text{(degenerate electron gas)}$$

$$E_F = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3} \quad \text{(nonrelativistic Fermi energy)}$$

$$r = r_\circ A^{1/3}, \qquad r_\circ = 1.2 \times 10^{-15} m \quad \text{(approximate average nuclear radius)}$$

$$\frac{n}{s} + \frac{n\prime}{s\prime} = \frac{n\prime - n}{R} \text{(refraction of paraxial rays)}$$

MASSES OF SOME ELEMENTARY PARTICLES

	Rest Mass, m_0 (kg)	m_0c^2 (MeV)
Electron	9.109×10^{-31}	0.511
Proton	1.673×10^{-27}	938.3
Neutron	1.675×10^{-27}	939.6
Atomic mass unit (1 amu)	1.661×10^{-27}	931.5

VISIBLE LIGHT SPECTRUM

300	400	500		600		700(Nanometer)	
← Ultraviolet	Violet	Blue	Green	Yellow	Orange	Red	Infrared →

PART A: Obligatory Problems

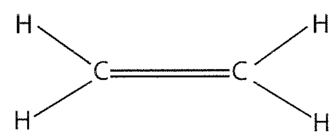
1. (Quantum Mechanics) Consider an electron residing in a one-dimensional potential well, where the potential well has the form $V \to \infty$ for all $x \leq 0$ and $V(x) = V_0 x/a$ for all positive $x > 0$. Use the variational method to estimate the ground state energy of the system.

2. (Quantum Mechanics) An electron is in a state with the z-component of the spin angular momentum being $+\hbar/2$. The electron is introduced into an apparatus that measures the component of the spin angular momentum along an arbitrary direction vector $\hat{\mathbf{n}}$. What is the probability of observing a spin angular momentum of $+\hbar/2$ along $\hat{\mathbf{n}}$?

$\sin^2\left(\frac{x}{2}\right) = \frac{1}{2}(1 - \cos x)$

$\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) = \frac{1}{2}\sin x$

3. (Quantum Mechanics) An ethylene molecule C_2H_4, has a structure as shown below.



The pair of hydrogen atoms on either side can rotate around the axis formed by the two Carbon atoms so that it has a relative angle ϕ, with respect to the other pair. The potential induced by this rotation can be approximated by

$$V(\phi) = V_0(1 - \cos\phi), 0 < \phi \leq 2\pi$$

(a) For sufficiently small ϕ, one can approximate the cosine by the first two terms in the power series expansion. Using this approximation, and neglecting tunneling between the two minima, find the energy of the (degenerate) ground states, corresponding to oscillations around the minima.

(b) Now consider the effect of the next term in the power series expansion. Using first-order perturbation theory, compute the effect on the energy of the ground state(s).

(c) Now consider the effect of quantum tunneling. This allows a small amplitude for transitions between states localized to the two minima. Explain how the presence of this tunneling will effect the spectrum of eigenstates. Is this effect larger or smaller for the higher energy states?

4. (Statistical Mechanics) A motionless spin-1 particle sits in a uniform magnetic field B, which, without loss of generality, can be assumed to be oriented in the z direction. The z component of magnetic moment of the particle is $g\mu_B s_z$, where μ_B is the Bohr magneton, g is a g-factor depending on the nature of the particle, and s_z is the z component of the spin.

   (a) Find an expression for (the magnetic part of) the partition function of the particle when it is in thermal equilibrium at temperature T.

   (b) Hence or otherwise find an expression for the thermal average magnetic moment of the particle.

5. (Statistical Mechanics) The density of electromagnetic modes in a cavity of volume V is

$$D(\omega) = \frac{V\omega^2}{\pi^2 c^3},$$

where ω is (angular) frequency and c is the speed of light.

(a) Find an expression for the energy density of electromagnetic radiation with frequency between ω and $\omega + d\omega$ in such a cavity at temperature T.

(b) Explain how this result can be used to derive the frequency variation of the intensity of radiation from a black body, and hence show that the black body spectrum is of the form

$$I = I_0 \frac{\omega^3}{\exp(\hbar\omega/k_B T) - 1},$$

where I_0 is a constant (which you don't need to calculate).

(c) Find an expression for the frequency at which this spectrum peaks. You will probably need to know that the solution to the equation $3e^{-x} + x - 3 = 0$ is $x = 2.821\ldots$

(d) The cosmic black body background has a temperature of about 2.7K. Use the result of part (c) to calculate at what frequency (in Hertz) the peak background radiation occurs. What part of the electromagnetic spectrum does this peak fall in?

6. (Statistical Mechanics) Estimate a bound, in K, on the temperatures at which electrons in a metal with 10^{22} electrons/cm^3 could be treated using Maxwell-Boltzmann statistics. Assume the effective mass of electrons is the same as that of free electrons and the melting temperature of the metal is infinite.

7. (Atomic Physics) A repulsive central potential has the form $V(r) = A/r^2$. Find the differential scattering cross section of the potential $V(r)$ for a particle of mass m in the Born approximation. Express your result as a function of the scattering angle, θ, and the magnitude of the wave-vector, k.

$\int_0^\infty \frac{\sin ax}{x} dx = \frac{\pi}{2}$ for positive a.

8. (Nuclear Physics) In an experiment on Compton scattering, stationary electrons are bombarded by photons whose energy is equal to the rest energy of an electron. For events in which the scattered photon and the recoil electron have momenta of the same magnitude, find the angle between them. What is the momentum and speed of the recoil electron in this case?

PART B: Optional Problems

9. (Quantum Mechanics) The spin interaction of a spin-1 ion in a crystal is modeled by the following Hamiltonian:

$$H = \frac{a}{\hbar^2} S_z^2 + \frac{b}{\hbar^2}(S_x^2 + S_y^2)$$

Assume $b << a$. Find the energy levels to leading non-vanishing order in b. Useful equations:

$$S_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad S_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad S_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

10. (Atomic Physics) Consider an atom that follows the Russell-Saunders (or L-S) coupling scheme. The atom has a ground state 6 $^2S_{1/2}$ and an excited state 6 $^2P_{3/2}$. We ignore hyperfine effects. The wavelength of a laser beam driving the transition between these states is $\lambda = 852.356$ nm.

(a) What are the values of the intrinsic electron spin S, the orbital electron angular momentum L, and the combined angular momentum J for both the ground and excited state?

(b) A magnetic field of $B = 3$ Tesla is applied. Draw a level diagram that shows the Zeeman splitting of both the ground and the excited state, with the correct number of Zeeman-split magnetic sub-levels. Assign proper values of the magnetic quantum number to the sub-levels in your drawing.

(c) Determine the Lande g-factors of the ground and excited level.

(d) What is the wavelength of the transition 6 $^2S_{1/2}\ |m_J = 1/2\rangle \rightarrow 6\ ^2P_{3/2}\ |m_J = 3/2\rangle$ in the magnetic field?

The Bohr magneton is $\mu_B = 9.2740 \times 10^{-24}$ Am^2. 1 Tesla $= 10^4$ Gauss.

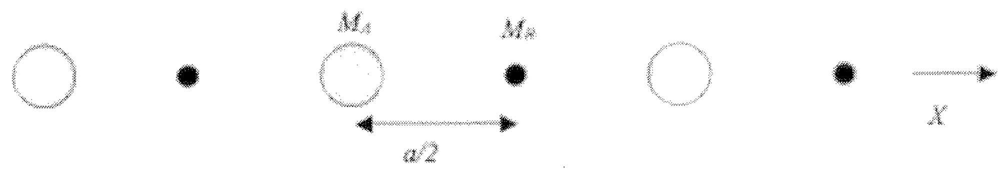
11. (Particle Physics) The top quark was discovered at Fermilab through the process

$$p + \bar{p} \to t\bar{t} \to b\bar{b}W^+W^-.$$

(a) In the frame where the top quark is at rest, find the energy of the b quark in the top quark decay. (You may neglect its mass) Keep the mass of the W, m_W.

(b) Estimate the fraction of events where both W bosons decay to light leptons ($l = e^\pm$ or $\mu^\pm$). You may negelect the masses of all fermions (except for the top quark), and approximate the the CKM matrix as completely diagonal.

12. (Condensed Matter) Consider a one-dimensional lattice with a unit cell consisting of two atoms with masses M_A and M_B. The equilibrium spacing between the atoms is $a/2$ and they are only allowed to move in x-direction. The neighboring atoms can viewed as if they are connected to each other by springs with a spring constant K.



(a) Sketch the phonon dispersion curve for such a diatomic linear lattice.

(b) Calculate the speed of sound for this lattice.

(c) What is the energy of the highest energy phonon mode?