Name: ______________________ Name-code: ______________________

University of Michigan Physics Department
Graduate Qualifying Examination

Part II – Modern Physics
Saturday, May 13, 2006 9:00am-1:00pm

Please write your name-code on each page of the exam.

This is a closed book exam – but you may use the "Constants, Conversions, and Formulas" sheet we provide for this exam.

Show your work in an organized manner to receive partial for it. If you need to make an assumption or an estimate, indicate it clearly.

You must answer the first 8 obligatory questions and two of the optional four questions. Indicate which of the latter you wish us to grade (e.g., circle the question number). We will only grade the indicated optional questions.

Please do not write on this page. The hatched column to the right and the total score will be filled in by the graders.

Good Luck!

Total Score:

1	
2	
3	
4	
5	
6	
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11	
12	

"Constants, Conversions, and Formulas"

University of Michigan Physics Department Graduate Qualifying Examination

SOME FUNDAMENTAL CONSTANTS IN CONVENIENT UNITS

$$
\begin{aligned}
\text{speed of light} \quad c &= 2.998 \times 10^{8}\ \text{m/s} \\
\text{electron charge} \quad e &= 1.602 \times 10^{-19}\ \text{C} \\
\text{Planck's constant} \quad h &= 6.626 \times 10^{-34}\ \text{J}\cdot\text{s} = 4.136 \times 10^{-15}\ \text{eV}\cdot\text{c} \\
\hbar = h/2\pi &= 1.055 \times 10^{-34}\ \ \text{J}\cdot\text{s} = 0.658 \times 10^{-15}\ \text{eV}\cdot\text{s} \\
\text{Rydberg constant} \quad R_{\infty} &= 1.097 \times 10^{6}\ \text{m}^{-1} \\
\text{Coulomb constant} \quad k = (4\pi\epsilon_0)^{-1} &= 8.988 \times 10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2 \\
\text{Universal gas constant} \quad R &= 8.3\text{J/K}\cdot\text{mol} \\
\text{Avogadro's number} \quad N_A &= 6 \times 10^{23}\ \text{mol}^{-1} \\
\text{Boltzmann's constant} \quad k_B = R/N_A &= 1.38 \times 10^{-23} J/K = 8.617 \times 10^{-5}\text{eV/K} \\
\text{Stefan} - \text{Boltzmann constant} \quad \sigma &= 5.6703 \times 10^{-8}\ \text{W/m}^2\text{K}^4 \\
\text{radius of the sun} \quad R_{sun} &= 6.96 \times 10^{8}\ \text{m} \\
\text{radius of the moon} \quad R_{moon} &= 1.74 \times 10^{6}\ \text{m} \\
\text{radius of the earth} \quad R_{earth} &= 6.37 \times 10^{6}\ \text{m} \\
G_N &= 6.67 \times 10^{-11} m^3/kg/s^2 = 6.71 \times 10^{-39} GeV^{-2}
\end{aligned}
$$

SOME USEFUL CONVERSIONS AND COMBINATIONS

$$
\begin{aligned}
\text{fine structure constant} \quad \alpha &= ke^2/\hbar c = 1/137 \\
\text{Bohr magneton} \quad e\hbar/2m_e &= 9.27 \times 10^{-24}\text{J/T} = 5.79 \times 10^{-5}\text{eV/T} \\
hc &= 19.865 \times 10^{-26}\ \text{J}\cdot\text{m} = 12.41 \times 10^{3}\ \text{eV}\cdot\text{Å} = 1241\ \text{MeV}\cdot\text{fm} \\
\hbar c &= 3.165 \times 10^{-26}\ \text{J}\cdot\text{m} = 1973\ \text{eV}\cdot\text{Å} = 197.3\ \text{MeV}\cdot\text{fm} \\
ke^2 &= 1.44\ \text{MeV}\cdot\text{fm} \\
& 1\text{Å} = 10^{-10}\ \text{m} = 10^{5}\ \text{fm} \qquad 1\text{eV} = 1.602 \times 10^{-19}\ \text{J}
\end{aligned}
$$

SOME USEFUL RELATIONS

$$\frac{C_V}{Nk} = 9\,(T/\theta_D)^3 \int_0^{\theta_D/T} \frac{x^4 e^4 dx}{(e^x - 1)^2} \quad \text{(Debye formula)}$$

$$= 3\left[1 - \frac{1}{20}\left(\frac{\theta_D}{T}\right)^2 + \ldots\right] = \frac{12\pi^4}{5}\left(\frac{T}{\theta_D}\right)^3 (1 + \ldots)$$

$$\frac{U}{N} = \frac{3}{5}E_F\left[1 + \frac{5\pi^2}{12}\left(\frac{kT}{E_F}\right)^2 - \frac{\pi^4}{16}\left(\frac{kT}{E_F}\right)^4 + \ldots\right] \quad \text{(degenerate electron gas)}$$

$$E_F = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3} \quad \text{(nonrelativistic Fermi energy)}$$

$$r = r_\circ A^{1/3}, \qquad r_\circ = 1.2 \times 10^{-15} m \quad \text{(approximate average nuclear radius)}$$

$$E_b = C_1 A - C_2 A^{2/3} - C_3 \frac{Z(Z-1)}{A^{1/3}} - C_4 \frac{(N-Z)^2}{A} \quad \text{(semiempirical binding energy of a nucleus)}$$

$$\frac{n}{s} + \frac{n\prime}{s\prime} = \frac{n\prime - n}{R} \text{(refraction of paraxial rays)}$$

MASSES OF SOME ELEMENTARY PARTICLES

	Rest Mass, m_0 (kg)	m_0c^2 (MeV)
Electron	9.109×10^{-31}	0.511
Proton	1.673×10^{-27}	938.3
Neutron	1.675×10^{-27}	939.6
Atomic mass unit (1 amu)	1.661×10^{-27}	931.5

VISIBLE LIGHT SPECTRUM

300	400	500		600		700(Nanometer)	
← Ultraviolet	Violet	Blue	Green	Yellow	Orange	Red	Infrared →

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Part A: Obligatory Problems

1. Two spin-1/2 particles interact with an external magnetic field $\vec{B}$ through

$$H_{ext} = -\alpha(\vec{S}_1 + \vec{S}_2) \cdot \vec{B}$$

where α is a constant parametrizing the magnetic moment of the particles. The anti-ferromagnetic interaction

$$H_{int} = J\vec{S}_1 \cdot \vec{S}_2$$

couples the two particles. Neglecting all other terms in the Hamiltonian, find the spectrum of the system.

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2. The Hamiltonian for two identical noninteracting particles in the infinite square well potential is of the form

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x_1,x_2)}{dx_1^2} - \frac{\hbar^2}{2m}\frac{d^2\psi(x_1,x_2)}{dx_2^2} = E\psi(x_1,x_2)$$

for $0 \le x_1, x_2 \le a$. Consider the following wave function

$$\psi(x_1,x_2) = \frac{\sqrt{2}}{a}\left(\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right) - \sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)\right).$$

(a) What kind of indistinguishable particles are described by this wave function?
(b) What is the energy of this state?
(c) Is this the ground state of the described system?

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3. (a) A He^+-ion undergoes a spontaneous electromagnetic decay from $|n = 6, l = 1, m = 1, m_s = 1/2\rangle$ to $|n = 5, l = 0, m = 0, m_s = 1/2\rangle$ in a field of 10^3 Tesla. Calculate the wavelength of the emitted photon.
(b) Under what conditions of (a) would you expect significant spontaneous decay into states with $m_s = -1/2$? Explain your answer.

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4. In graphite the carbon atoms are strongly bound in *layers* which are weakly coupled to each other. Predict the low-temperature lattice contribution to the heat capacity of graphite. It is *not* proportional to T^3, but to another power of T.

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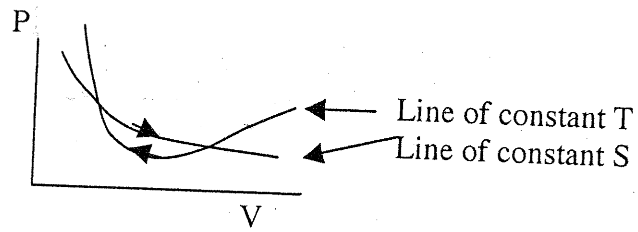
5. An engineer (who used to work at Enron) asks you to invest in an invention: it is a machine which works on the cycle shown below in the P, V space. Assume all changes are slow, and the system remains in equilibrium.

   Show that if there was a substance like this attached to a heat reservoir and a system of pulleys, the machine would decrease the total entropy. Is this an engine or a refrigerator? Would you invest?



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6. When a system interacts with a heat reservoir, its energy fluctuates with dispersion measured by the variance $\overline{(\Delta E)^2}$. Suppose the system is quantum mechanical, with energy eigenvalues given by $\epsilon, 0$ (with degeneracy 2), and $-\epsilon$.
(a) Compute the dispersion of the energy.
(b) Verify that the limits as $T \to 0$ and $T \to \infty$ agree with qualitative arguments.

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7. The spin part of the wave function for a proton is

$$
\begin{aligned}
|p^\uparrow\rangle \quad = \quad & \frac{1}{\sqrt{18}}\Big\{|u^\uparrow u^\uparrow d^\downarrow\rangle + 2|u^\uparrow d^\downarrow u^\uparrow + 2|d^\downarrow u^\uparrow u^\uparrow\rangle \\
& -|u^\uparrow u^\downarrow d^\uparrow\rangle - |u^\uparrow d^\uparrow u^\downarrow\rangle - |d^\uparrow u^\uparrow u^\downarrow\rangle \\
& -|u^\downarrow u^\uparrow d^\uparrow\rangle - |u^\downarrow d^\uparrow u^\uparrow\rangle - |d^\uparrow u^\downarrow u^\uparrow\rangle\Big\},
\end{aligned}
$$

and a similar expression for the neutron. Let us assume that the constituent quark model is correct and that the magnetic moment of the proton is given by

$$\vec{\mu}_p = \vec{\mu}_u + \vec{\mu}_u + \vec{\mu}_d, \quad \text{where}$$

$$\vec{\mu}_{u,d} = \mu_{u,d}\vec{\sigma}_{u,d}, \quad \mu_{u,d} = \frac{z_{u,d}e\hbar}{2m_{u,d}c}$$

and $z_u = 2/3$, $z_d = -1/3$, with $\vec{\sigma}$ being the Pauli matrices.

(a) Find the magnetic moments of the nucleons $\mu_{p,n}$ in terms of $\mu_{u,d}$.
(b) Now, assume that $m_u = m_d$. What is the ratio of $\frac{\mu_p}{\mu_n}$?
(c) What is the approximate value in MeV of $m_u \simeq m_d$?
(d) The total wave function for a nucleon must be anti-symmetric under the exchange of any pair of quarks. What must also be considered to make this so?

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8. One of the successful applications of the shell model is to a nucleus with only one left-over particle (or hole). The other nucleons are in closed shells. Consider the magnetic moment of such a nucleus, which is given by

$$\vec{\mu}^{op} = \frac{\mu_n}{\hbar}(g_l \vec{L} + g_s \vec{S}),$$

where $g_l = 1(0)$, and $g_s = 5.58(-3.81)$ for a proton (neutron), and $\mu_n = \frac{e\hbar}{2m_n c}$, m_n being the mass of a nucleon. $\vec{L}$ is the orbital momentum operator and $\vec{S}$ is the spin of the nucleon. The nuclear magnetic moment is then the expectation value

$$\vec{\mu}_{nucleus} = \langle \psi_{nucleus} | \vec{\mu}^{op} | \psi_{nucleus} \rangle. \qquad (1)$$

Then,

$$\vec{\mu}_{nucleus} = C \langle \psi_{nucleus} | \vec{J} | \psi_{nucleus} \rangle, \qquad (2)$$

where $\vec{J} = \vec{L} + \vec{S}$, and C is a constant to be determined.
(a) Why is eq.(1) above a good approximation?
(b) Why is eq.(2) above a correct statement?
(c) Let the eigenvalue of the total angular momentum of the nucleus be j and the magnetic quantum number be m. It can be shown that the constant C is given by

$$C = \frac{\langle jm | \vec{J} \cdot \vec{\mu}^{op} | jm \rangle}{\langle jm | \vec{J} \cdot \vec{J} | jm \rangle}.$$

Calculate C in terms of g_l, g_s, l, s, j, and m.

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Part B: Optional Problems

9. The ground state wave function of a hydrogen atom is

$$\psi(\vec{r}) = \frac{1}{\sqrt{\pi}} r_0^{-3/2} e^{-r/r_0}$$

What is the corresponding probability distribution of *momenta*?

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10. Zeeman-slowing of rubidium atoms.

A laser beam (wavelength $\lambda = 780\,\mathrm{nm}$) counter-propagates an atomic beam (atomic mass $87u$). The atomic beam propagates in the $+z$-direction. Under resonant conditions, the atoms scatter photons at a rate of $15 \times 10^6\,s^{-1}$.

(a) Assuming that the laser is always on-resonance, determine the average acceleration that the atoms experience, the time it takes to slow an atom to zero velocity, and the length of the slowing trajectory. Assume that the atoms have an initial velocity of $200\,m/s$.

(b) Assuming that the slowing starts at location $z = 0$, determine the Doppler shift of the atomic transition as a function of position.

(c) A magnetic field $\mathbf{B}(z) = \hat{\mathbf{z}}B(z)$ is used to maintain the resonance condition (i.e., to compensate the change in Doppler shift). The magnetic field $B(z)$ gradually increases from zero at $z = 0$ to some maximum value, and causes a position-dependent Zeeman shift of the frequency of the slowing transition of $-\mu_B B(z)$. The laser is tuned such that it is initially, at $z = 0$, in resonance with the atoms moving at $200m/s$. Determine the function $B(z)$ for which the atoms will remain in resonance with the laser beam throughout the length of the slowing trajectory. What is the magnetic field at the location where the atoms reach zero velocity?

Note: This version of a Zeeman slower is called a σ^--slower.

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11. In the collision of $e^+ + e^-$, where the total energy in the center of mass system is close to the mass of Z^0, we can have hadrons, lepton pairs $(e^+e^-, \mu^+\mu^-, \tau^+\tau^-)$ and neutrino pairs as the final productions. The cross-section for producing hadrons near the Z mass is given by

$$\sigma_0 = 12\pi \frac{\Gamma_{lepton}\Gamma_{hadrons}}{M_Z^2 \Gamma_{total}^2},$$

where Γ_{lepton} is the width for a Z^0 to decay into each flavor of lepton pair, $\Gamma_{hadrons}$ is that into all hadrons and Γ_{total} is the total width. Measurements give

$$\sigma_0 \simeq 41.49\,\text{nb}, \quad R_l \simeq \frac{\Gamma_{hadrons}}{\Gamma_{lepton}} \simeq 20.77, \qquad M_Z \simeq 91.182\,\text{GeV},$$

We also have the theoretical result that for each flavor of neutrino pair $\Gamma_{\nu\bar{\nu}}/\Gamma_{lepton} = 1.99$.

(a) From the data given above, can you estimate how many flavors of neutrinos could be produced when the center of mass energy is near the Z-mass? (The conversion factor is $1(\text{GeV})^{-2} \simeq 0.389\,\text{mb}$.)

(b) Are very massive neutrinos excluded by the data? Why?

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12. Consider a two-dimensional metal with a square crystal structure with lattice constant a. There are two electrons per unit cell.
(a) Find the reciprocal lattice and sketch the Brillouin zone.
(b) Figure out the radius of the Fermi disc in terms of a. Sketch the Fermi disc on top of the Brillouin zone, as well as you can.